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Governing Equations of a Stiffened Laminated Inhomogeneous Conical Shell

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This study presents the dynamic equations of a stiffened composite laminated conical thin shell under the influence of initial stresses. The governing equations of a truncated conical shell are based on the Donnell–Mushtari theory of thin shells including the transverse shear deformation and rotary inertia. The extension-bending coupling is considered in the derivation. The composite laminated conical shell is also reinforced at uniform intervals by elastic rings and/or stringers. The stiffening elements are relatively closely spaced, and therefore the stiffeners are smeared out along the conical shell. The inhomogeneity of material properties because of temperature, moisture, or manufacturing processes is taken into account in the constitutive equations. A generalized variational theorem is derived so as to describe the complete set of the fundamental equations of the conical shell. Next, the uniqueness is examined in solutions of the dynamic equations of the conical shell, and the boundary and initial conditions are shown to be sufficient for the uniqueness in solutions. The equations of the laminated composite conical shell are solved by the use of the finite difference method as an illustrative example. The accuracy of results is tested by certain earlier results, and a good agreement is found.

Introduction

A STIFFENED laminated conical shell is one of the common structural elements used in modern airplane, missile, booster, and other space vehicles. The dynamic behavior of stiffened laminated conical shells under the dynamic loads and initial stresses is of considerable engineering importance for determination of the failure and fatigue life of the shell. The inhomogeneity as a result of environmental effects such as temperature and moisture may degrade the mechanical properties of anisotropic materials and affect the dynamic behavior of the shell and should be taken into account in the constitutive equations of the conical shell.

Studies of the vibrations of cylindrical and truncated conical shells made of isotropic and anisotropic materials are referenced in Ref. 1. Although much literature exists on the free vibration of isotropic conical shells, e.g., Refs. 2–5, fewer studies were found about stiffened, orthotropic, or anisotropic conical shells. Singer[6] and Weingarten[7] derived the equations of motion of a Donnell–Mushtari-type orthotropic shell theory. Bacon and Bert[8] showed the effect of changing the ratio of orthotropic constants upon the fundamental frequencies of shells of revolution. Cohen[9] investigated the asymmetrical free vibrations of ring-stiffened orthotropic shells of revolution. Mecitoglu and Dökmeci[10] developed a shell finite element that includes smeared stringers and rings and examined the uniqueness on the solutions. Weingarten,[11] Goldberg et al.,[12,13] and Schneider et al.[14] studied the vibration of the conical shells subjected to initial stresses. Newton[15] obtained results for the conical shells having orthotropic material properties that vary in the meridional direction. Martin[16] studied the free vibrations of anisotropic conical shells. Heyliger and Jilani[17] considered the inhomogeneity in the free vibrations of elastic layered cylinders and spheres. Inhomogeneity of the material properties as a result of temperature was studied by Mecitoglu[18] for a truncated isotropic conical shell. Sivadas and Ganesan,[19] and Sankaranarayanan et al.[20] studied the laminated conical shells with variable thickness. Kayran and Vinson[21] examined the effect of transverse shear and rotary inertia on the free vibrations of truncated conical shells. Tong studied the free vibration of orthotropic[22] and laminated[23,24] conical shells using a simple and exact solution technique. Langley[25] applied a dynamic stiffness technique to the vibration of the stiffened shell structures. Ley et al.[26] examined the buckling of the stiffened anisotropic conical shells. Although the uniqueness in solutions of the elastodynamic problems is very important, only a few works examining this subject are found in the literature.[27]

The purpose of this paper is to derive all of the dynamic equations of a composite laminated conical thin shell reinforced by stringers and rings and to examine the uniqueness in solutions of the governing shell equations. The dynamic equations of a stiffened composite laminated conical shell are derived within the frame of the Donnell–Mushtari theory of elastic thin shells. A generalized variational theorem is given so as to describe the complete set of the fundamental equations of the conical shell. The geometric nonlinearities and initial stresses are taken into account in the derivation of governing equations. The rings and/or stringers are smeared out along the conical shell. The inhomogeneity of material properties as a result of temperature, moisture, or manufacturing processes is taken into account in the constitutive equations. The uniqueness is examined in solutions of the dynamic governing equations of stiffened shells, and a theorem of uniqueness is given that enumerates the initial and boundary conditions sufficient for the uniqueness. The dynamic equations of the stiffened laminated conical shell are solved by the finite difference method to obtain the vibration characteristics, and a good agreement is obtained with certain earlier results.

Governing Equations

Consider a conical shell as shown in Fig. 1, where R indicates the radius of the cone at the large end, α denotes the semivertex angle of the cone, and L is the cone length along its generator. The terms h_S and h_R are the height of the stringers and rings, respectively; b_S and b_R are the width of the stringers and rings, respectively; Θ is the angular spacing of the stringers; and S is the spacing of the rings. We use the s–θ coordinate system; s is measured along the cone's generator starting at the cone vertex, and θ is the circumferential coordinate.

Displacement Field

The Weierstrass theorem states that any function that is continuous in an interval may be approximated uniformly by polynomials in this interval. Thus, the displacement field in the shell can be represented by the following relationships:

$$
\begin{aligned}
U(s,\theta,z) &= u(s,\theta) + z\beta_s(s,\theta) + z^2\gamma_s(s,\theta) + \cdots \\
V(s,\theta,z) &= v(s,\theta) + z\beta_\theta(s,\theta) + z^2\gamma_\theta(s,\theta) + \cdots \\
W(s,\theta,z) &= w(s,\theta) + z\beta_z(s,\theta) + z^2\gamma_z(s,\theta) + \cdots
\end{aligned}
\qquad (1)
$$

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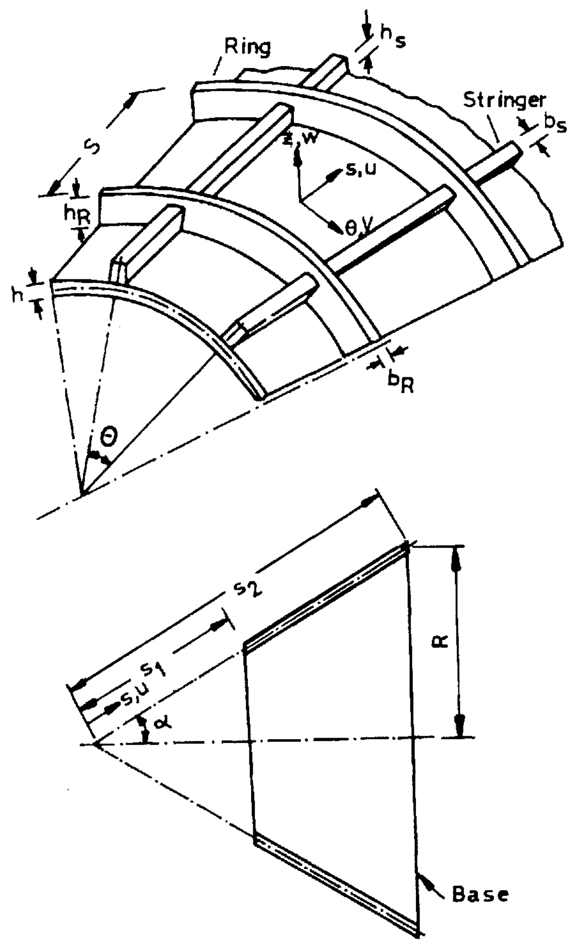

Fig. 1 Geometry and coordinates.

where U, V, and W are the displacement components in the directions of axes s, θ, and z, respectively. The Kirchoff–Love hypothesis for linear elastic thin shells results in the linearly distributed tangential displacements and a constant normal displacement through the thickness of the shell, and hence Eqs. (1) are simplified as follows:

$$
\begin{aligned}
U(s,\theta,z) &= u(s,\theta) + z\beta_S(s,\theta)\\
V(s,\theta,z) &= v(s,\theta) + z\beta_\theta(s,\theta)\\
W(s,\theta,z) &= w(s,\theta)
\end{aligned}
\tag{2}
$$

where u, v, and w are the components of displacement at the middle surface in the s, θ, and normal directions, respectively, and β_s and β_θ are the rotations of the normal to the middle surface during deformation about the s and θ axes, respectively.

Kinematical Relations

Kinematical relations of a thin truncated conical shell can be approximated by the use of the preceding representation of the displacement field. The geometrical nonlinearities are taken into account in the formulation. Now, we define a functional J_k for the kinematical relations whose first variation is given by

$$
\begin{aligned}
\delta J_k = \int_T \mathrm{d}t \int\!\!\int_A \Bigg(& \left\{\varepsilon_s - \lambda\left[\frac{\partial u}{\partial\bar{s}} + \frac{\lambda}{2}\left(\frac{\partial w}{\partial\bar{s}}\right)^2\right]\right\}\delta N_s\\
& + \left\{\varepsilon_\theta - \lambda\left[\frac{u}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial v}{\partial\bar{\theta}} + \cot\alpha\frac{w}{\bar{s}} + \frac{\lambda}{2\bar{s}^2}\left(\frac{\partial w}{\partial\bar{\theta}}\right)^2\right]\right\}\delta N_\theta\\
& + \left[\varepsilon_{s\theta} - \lambda\left(\frac{1}{\bar{s}}\frac{\partial u}{\partial\bar{\theta}} + \frac{\partial v}{\partial\bar{s}} - \frac{v}{\bar{s}} + \frac{\lambda}{\bar{s}}\frac{\partial w}{\partial\bar{s}}\frac{\partial w}{\partial\bar{\theta}}\right)\right]\delta N_{s\theta}\\
& + \left(\kappa_s - \lambda\frac{\partial\beta_s}{\partial\bar{s}}\right)\delta M_s + \left[\kappa_\theta - \lambda\left(\frac{1}{\bar{s}}\frac{\partial\beta_\theta}{\partial\bar{\theta}} + \frac{\beta_s}{\bar{s}}\right)\right]\delta M_\theta\\
& + \left[\kappa_{s\theta} - \lambda\left(\frac{\partial\beta_\theta}{\partial\bar{s}} - \frac{\beta_\theta}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial\beta_s}{\partial\bar{\theta}}\right)\right]\delta M_{s\theta}\\
& + \left[\varepsilon_{sz} - \lambda\left(\frac{\beta_s}{\lambda} + \frac{\partial w}{\partial\bar{s}}\right)\right]\delta Q_s\\
& + \left[\varepsilon_{\theta z} - \lambda\left(\frac{\beta_\theta}{\lambda} - \cot\alpha\frac{v}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial w}{\partial\bar{\theta}}\right)\right]\delta Q_\theta\Bigg)\,\mathrm{d}A
\end{aligned}
\tag{3}
$$

where ε_s, ε_θ, and $\varepsilon_{s\theta}$ are the membrane strains of the middle surface, ε_{sz} and $\varepsilon_{\theta z}$ are the transverse shear strains, and κ_s, κ_θ, and $\kappa_{s\theta}$ are the bending strains. The parameter λ $(\sin\alpha/R)$ is a characteristic of the conical shell. The coordinates $\bar{s}$ and $\bar{\theta}$ are defined as $\bar{s} = s\lambda$ and $\bar{\theta} = \theta\sin\alpha$, respectively. The strain-displacement equations of the Donnell–Mushtari shell theory are obtained vanishing the first variation of functional.

Constitutive Equations

In the plate and shell theory, it is convenient to introduce the force and moment resultants by integrating the stresses over the shell thickness. The constitutive equations of an anisotropic material relate the force and moment resultants to the membrane and bending strains. Here, the bending-stretching coupling is considered in the constitutive equations. The stiffening elements are relatively closely spaced, and therefore the stiffeners are smeared out along the conical shell. The material properties of the laminated conical shell are changed to take into account the smeared out stiffeners. Hence, the constitutive equations of the stiffened laminated conical shell with stretching-bending coupling are given in variational form as

$$
\begin{aligned}
\delta I_c = \int_T \mathrm{d}t \int_A \{ & [N_s - (A_{11}\varepsilon_s + A_{12}\varepsilon_\theta + B_{11}\kappa_s + B_{12}\kappa_\theta)]\delta\varepsilon_s\\
& + [N_\theta - (A_{12}\varepsilon_s + A_{22}\varepsilon_\theta + B_{12}\kappa_s + B_{22}\kappa_\theta)]\delta\varepsilon_\theta\\
& + [N_{s\theta} - (A_{33}\varepsilon_{s\theta} + B_{33}\kappa_{s\theta})]\delta\varepsilon_{s\theta}\\
& + [M_s - (B_{11}\varepsilon_s + B_{12}\varepsilon_\theta + D_{11}\kappa_s + D_{12}\kappa_\theta)]\delta\kappa_s\\
& + [M_\theta - (B_{12}\varepsilon_s + B_{22}\varepsilon_\theta + D_{12}\kappa_s + D_{22}\kappa_\theta)]\delta\kappa_\theta\\
& + [M_{s\theta} - (B_{33}\varepsilon_{s\theta} + D_{33}\kappa_{s\theta})]\delta\kappa_{s\theta}\\
& + [Q_s - A_{55}\varepsilon_{sz}]\delta\varepsilon_{sz} + [Q_\theta - A_{44}\varepsilon_{\theta z}]\delta\varepsilon_{\theta z}\}\,\mathrm{d}A = 0
\end{aligned}
\tag{4}
$$

where

$$
\begin{aligned}
A_{ij} &= \sum_{k=1}^{N_L} \bar{Q}_{ij}^{(k)}(h_k - h_{k-1}) + E_{ij}\\
B_{ij} &= \frac{1}{2}\sum_{k=1}^{N_L} \bar{Q}_{ij}^{(k)}\left(h_k^2 - h_{k-1}^2\right) - F_{ij}\\
D_{ij} &= \frac{1}{3}\sum_{k=1}^{N_L} \bar{Q}_{ij}^{(k)}\left(h_k^3 - h_{k-1}^3\right) + G_{ij} \qquad i,j = 1,2,3\\
\{A_{44}, A_{55}\} &= \sum_{k=1}^{N_L}\int_{h_{k-1}}^{h_k} \left\{\bar{Q}_{44}^{(k)}, \bar{Q}_{55}^{(k)}\right\} f(z)\,\mathrm{d}z
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
\{E_{11}, E_{22}, E_{12}, E_{33}\} &= \lambda\left\{\frac{E_S A_S}{\Theta}, \frac{E_R A_R}{\bar{S}}, 0, 0\right\}\\
\{F_{11}, F_{22}, F_{12}, F_{33}\} &= 2\lambda\left\{\frac{E_S A_S e_S}{\Theta}, \frac{E_R A_R e_R}{\bar{S}}, 0, 0\right\}\\
\{G_{11}, G_{22}, G_{12}, G_{33}\} &= \\
\lambda\Bigg\{\frac{E_S A_S\left(i_{\eta S}^2 + e_S^2\right)}{\Theta}, & \frac{E_R A_R\left(i_{\xi R}^2 + e_R^2\right)}{\bar{S}}, 0, \left(\frac{G_S J_S}{\Theta} + \frac{G_R J_R}{\bar{S}}\right)\Bigg\}
\end{aligned}
\tag{6}
$$

and

$$f(z) = \tfrac{5}{4}[1 - 4(z/h)^2] \tag{7}$$

where h is the total wall thickness of the conical shell. The terms E_S and $G_S J_S$ are the modulus of elasticity and torsional stiffness of a stringer, respectively; A_S is the cross-sectional area of stringer; i_S is the radius of gyration of the stringer cross-sectional area about a centroidal axis parallel to the θ axis; and e_S is the distance to the centroid of the stringer cross section from the shell middle surface (eccentricity). The subscript R indicates the properties of a ring. In Eqs. (6), $\bar{\Theta} = \Theta \sin\alpha$ and $\bar{S} = S\lambda$.

Elastic constants or material density may have a spatial change because of environmental causes such as temperature, moisture effect, or manufacturing processes. Therefore, the material of the conical shell is inhomogeneous in nature, and it should be considered in the constitutive relations. For purposes of demonstration, a simple linear variation in the Young modulus through laminates of the conical shell is considered,[17] described by

$$\begin{aligned} E_s(z) &= E_{s_0}\{1 + \alpha_s[1 + 2(z/h)]\} \\ E_\theta(z) &= E_{\theta_0}\{1 + \alpha_\theta[1 + 2(z/h)]\} \end{aligned} \tag{8}$$

where α_s and α_θ are the parameters and E_{s_0} and E_{θ_0} are the Young modules of the inner surface of truncated conical shell.

Equilibrium Equations

Now, the equilibrium equations and the boundary and initial conditions for the vibration of a stiffened composite conical shell with initial stresses are derived from the virtual work principle. The rotary inertia of the conical shell and stiffeners is taken into account. Thus the virtual work principle is applied to the stiffened composite conical shell under the influence of the initial stresses and the following equation is obtained:

$$\begin{aligned} \delta J_e = \int_T \mathrm{d}t \iint_A & \big[(N_s + N_s^i)\delta\varepsilon_s + (N_\theta + N_\theta^i)\delta\varepsilon_\theta \\ & + (N_{s\theta} + N_{s\theta}^i)\delta\varepsilon_{s\theta} + (M_s + M_s^i)\delta\kappa_s + (M_\theta + M_\theta^i)\delta\kappa_\theta \\ & + (M_{s\theta} + M_{s\theta}^i)\delta\kappa_{s\theta}(Q_s + Q_s^i)\delta\varepsilon_{sz} + (Q_\theta + Q_\theta^i)\delta\varepsilon_{\theta z}\big]\,\mathrm{d}A \\ & - \iint_A \big[(q_s + q_s^i)\delta u + (q_\theta + q_\theta^i)\delta v + (q_z + q_z^i)\delta w \\ & + (m_s + m_s^i)\delta\beta_s + (m_\theta + m_\theta^i)\delta\beta_\theta\big]\,\mathrm{d}A \\ & - \iint_A \bigg[m_1(\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w}) + \frac{m_2}{2}(\dot{u}\delta\dot{\beta}_s + \dot{\beta}_s\delta\dot{u} \\ & + \dot{v}\delta\dot{\beta}_\theta + \dot{\beta}_\theta\delta\dot{v}) + m_{12}\dot{\beta}_s\delta\dot{\beta}_s + m_{21}\dot{\beta}_\theta\delta\dot{\beta}_\theta\bigg]\,\mathrm{d}A = 0 \end{aligned} \tag{9}$$

where the superscript i indicates the initial stresses and the dot above the symbols denotes the derivation with respect to time. The terms q_s^i, q_θ^i, q_z^i, m_s^i, and m_θ^i are the static loads that cause the initial stresses, and q_s, q_θ, q_z, m_s, and m_θ are the dynamic loads. In Eq. (9) the parameters related with the mass of the conical shell are

$$\begin{aligned} m_1 &= \rho_c h + \frac{\lambda\rho_S A_S}{\bar{\Theta}} + \frac{\lambda\rho_R A_R}{\bar{S}} \qquad m_2 = \frac{2\lambda\rho_S A_S e_S}{\bar{\Theta}} + \frac{2\lambda\rho_R A_R e_R}{\bar{S}} \\ m_{12} &= \frac{\rho_c h^3}{12} + \frac{\lambda\rho_S A_S}{\bar{\Theta}}(e_S^2 + i_S^2) + \frac{\lambda\rho_R A_R}{\bar{S}}(e_R^2 + i_{pR}^2) \\ m_{21} &= \frac{\rho_c h^3}{12} + \frac{\lambda\rho_S A_S}{\bar{\Theta}}(e_S^2 + i_{pS}^2) + \frac{\lambda\rho_R A_R}{\bar{S}}(e_R^2 + i_R^2) \end{aligned} \tag{10}$$

where ρ_c, ρ_S, and ρ_R are the material densities of the shell, stringer, and ring, respectively; and i_{pS} and i_{pR} are the polar radius of gyration of the stringer and the ring about the centroidal axes, respectively.

Integrating Eq. (9) by parts, one arrives at the following variational equation:

$$\begin{aligned} \delta J_e = \int_T \mathrm{d}t \iint_A \Bigg(& \bigg\{-\frac{\partial}{\partial\bar{s}}(N_s + N_s^i) - \frac{1}{\bar{s}}\big[(N_s + N_s^i) - (N_\theta + N_\theta^i)\big] \\ & - \frac{1}{\bar{s}}\frac{\partial}{\partial\bar{\theta}}(N_{s\theta} + N_{s\theta}^i) + \frac{m_1}{\lambda}\ddot{u} + \frac{m_2}{2\lambda}\ddot{\beta}_s - \frac{q_s + q_s^i}{\lambda}\bigg\}\delta u \\ & + \bigg[-\frac{\partial}{\partial\bar{s}}(N_{s\theta} + N_{s\theta}^i) - \frac{2}{\bar{s}}(N_{s\theta} + N_{s\theta_i}) - \frac{1}{\bar{s}}\frac{\partial}{\partial\bar{\theta}}(N_\theta + N_\theta^i) \\ & - \frac{\cot\alpha}{\bar{s}}(Q_\theta + Q_\theta^i) + \frac{m_1}{\lambda}\ddot{v} + \frac{m_2}{2\lambda}\ddot{\beta}_\theta - \frac{q_\theta + q_\theta^i}{\lambda}\bigg]\delta v \\ & + \bigg\{-\frac{\partial}{\partial\bar{s}}(Q_s + Q_s^i) - \frac{1}{\bar{s}}\frac{\partial}{\partial\bar{\theta}}(Q_\theta + Q_\theta^i) - \frac{1}{\bar{s}}(Q_s + Q_s^i) \\ & + \frac{\cot\alpha}{\bar{s}}(N_\theta + N_\theta^i) - \frac{\lambda}{\bar{s}}\frac{\partial}{\partial\bar{s}}\bigg[(N_s + N_s^i)\bar{s}\frac{\partial w}{\partial\bar{s}}\bigg] \\ & - \frac{\lambda}{\bar{s}^2}\frac{\partial}{\partial\bar{\theta}}\bigg[(N_\theta + N_\theta^i)\frac{\partial w}{\partial\bar{\theta}}\bigg] - \frac{\lambda}{\bar{s}}\frac{\partial}{\partial\bar{s}}\bigg[(N_{s\theta} + N_{s\theta}^i)\frac{\partial w}{\partial\bar{\theta}}\bigg] \\ & - \frac{\partial}{\partial\bar{\theta}}\bigg[(N_{s\theta} + N_{s\theta}^i)\frac{\partial w}{\partial\bar{s}}\bigg] + \frac{m_1}{\lambda}\ddot{w} - \frac{q_z + q_z^i}{\lambda}\bigg\}\delta w \\ & + \bigg\{-\frac{\partial}{\partial\bar{s}}(M_s + M_s^i) - \frac{1}{\bar{s}}\big[(M_s + M_s^i) - (M_\theta + M_\theta^i)\big] \\ & - \frac{1}{\bar{s}}\frac{\partial}{\partial\bar{\theta}}(M_{s\theta} + M_{s\theta}^i) + \frac{1}{\lambda}(Q_s + Q_s^i) + \frac{m_2}{2\lambda}\ddot{u} + \frac{m_{12}}{\lambda}\ddot{\beta}_s \\ & - \frac{m_s + m_s^i}{\lambda}\bigg\}\delta\beta_s + \bigg[-\frac{\partial}{\partial\bar{s}}(M_{s\theta} + M_{s\theta}^i) - \frac{2}{\bar{s}}(M_{s\theta} + M_{s\theta}^i) \\ & - \frac{1}{\bar{s}}\frac{\partial}{\partial\bar{\theta}}(M_\theta + M_\theta^i) + \frac{1}{\lambda}(Q_\theta + Q_\theta^i) + \frac{m_2}{2\lambda}\ddot{v} + \frac{m_{21}}{\lambda}\ddot{\beta}_\theta \\ & - \frac{m_\theta + m_\theta^i}{\lambda}\bigg]\delta\beta_\theta\Bigg)\,\mathrm{d}A = 0 \end{aligned} \tag{11}$$

The variational statement (11) may be divided into two parts as follows:

$$\delta J_e = \delta J_e^i + \delta J_e^d \tag{12}$$

where J_e^i defines a functional of the initial stresses as a function of static loads and J_e^d defines a functional related with the vibration behavior as a function of dynamic loads. We may write the parts of Eq. (12) as follows:

$$\begin{aligned} \delta J_e^i &= \int_T \mathrm{d}t \iint_A \big(\tau_s^i\delta u + \tau_\theta^i\delta v + \tau_z^i\delta w + \tau_{sz}^i\delta\beta_s + \tau_{\theta z}^i\delta\beta_\theta\big)\,\mathrm{d}A \\ \delta J_e^d &= \int_T \mathrm{d}t \iint_A \big(\tau_s\delta u + \tau_\theta\delta v + \tau_z\delta w + \tau_{sz}\delta\beta_s + \tau_\theta\delta\beta_\theta\big)\,\mathrm{d}A \end{aligned} \tag{13}$$

where

$$\begin{aligned} \tau_s^i &= -\frac{\partial N_s^i}{\partial\bar{s}} - \frac{1}{\bar{s}}(N_s^i - N_\theta^i) - \frac{1}{\bar{s}}\frac{\partial N_{s\theta}^i}{\partial\bar{\theta}} - \frac{q_s^i}{\lambda} = 0 \\ \tau_\theta^i &= -\frac{\partial N_{s\theta}^i}{\partial\bar{s}} - \frac{2}{\bar{s}}N_{s\theta}^i - \frac{1}{\bar{s}}\frac{\partial N_\theta^i}{\partial\bar{\theta}} - \frac{\cot\alpha}{\bar{s}}Q_\theta^i - \frac{q_\theta^i}{\lambda} = 0 \\ \tau_z^i &= -\frac{\partial Q_s^i}{\partial\bar{s}} - \frac{1}{\bar{s}}\frac{\partial Q_\theta^i}{\partial\bar{\theta}} - \frac{1}{\bar{s}}Q_s^i + \frac{\cot\alpha}{\bar{s}}N_\theta^i - \frac{q_z^i}{\lambda} = 0 \\ \tau_{sz}^i &= -\frac{\partial M_s^i}{\partial\bar{s}} - \frac{1}{\bar{s}}(M_s^i - M_\theta^i) - \frac{1}{\bar{s}}\frac{\partial M_{s\theta}^i}{\partial\theta} + \frac{1}{\lambda}Q_s^i - \frac{m_s^i}{\lambda} = 0 \\ \tau_{\theta z}^i &= -\frac{\partial M_{s\theta}^i}{\partial\bar{s}} - \frac{2}{\bar{s}}M_{s\theta}^i - \frac{1}{\bar{s}}\frac{\partial M_\theta^i}{\partial\theta} + \frac{1}{\lambda}Q_\theta^i - \frac{m_\theta^i}{\lambda} = 0 \end{aligned} \tag{14}$$

This set is usually solved by stress functions for the unknown initial stresses. Once these are known, the following vibration equations are solved:

$$\tau_s = -\frac{\partial N_s}{\partial \bar{s}} - \frac{1}{\bar{s}}\left(N_s - N_\theta\right) - \frac{1}{\bar{s}}\frac{\partial N_{s\theta}}{\partial \bar{\theta}} + \frac{m_1}{\lambda}\ddot{u} + \frac{m_2}{2\lambda}\ddot{\beta}_s - \frac{q_s}{\lambda} = 0$$

$$\tau_\theta = -\frac{\partial N_{s\theta}}{\partial \bar{s}} - \frac{2}{\bar{s}}N_{s\theta} - \frac{1}{\bar{s}}\frac{\partial N_\theta}{\partial \bar{\theta}} - \frac{\cot\alpha}{\bar{s}}Q_\theta + \frac{m_1}{\lambda}\ddot{v} + \frac{m_2}{2\lambda}\ddot{\beta}_\theta - \frac{q_\theta}{\lambda} = 0$$

$$\tau_z = -\frac{\partial Q_s}{\partial \bar{s}} - \frac{1}{\bar{s}}\frac{\partial Q_\theta}{\partial \bar{\theta}} - \frac{1}{\bar{s}}Q_s + \frac{\cot\alpha}{\bar{s}}N_\theta - \frac{\lambda}{\bar{s}}\frac{\partial}{\partial \bar{s}}\left(N_s^i \bar{s}\frac{\partial w}{\partial \bar{s}}\right)$$
$$- \frac{\lambda}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}\left(N_\theta^i\frac{\partial w}{\partial \bar{\theta}}\right) - \frac{\lambda}{\bar{s}}\frac{\partial}{\partial \bar{s}}\left(N_{s\theta}^i\frac{\partial w}{\partial \bar{\theta}}\right) - \frac{\partial}{\partial \bar{\theta}}\left(N_{s\theta}^i\frac{\partial w}{\partial \bar{s}}\right)$$
$$+ \frac{m_1}{\lambda}\ddot{w} - \frac{q_z}{\lambda} = 0 \qquad (15)$$

$$\tau_{sz} = -\frac{\partial M_s}{\partial \bar{s}} - \frac{1}{\bar{s}}\left(M_s - M_\theta\right) - \frac{1}{\bar{s}}\frac{\partial M_{s\theta}}{\partial \bar{\theta}} + \frac{1}{\lambda}Q_s + \frac{m_2}{2\lambda}\ddot{u}$$
$$+ \frac{m_{12}}{\lambda}\ddot{\beta}_s - \frac{m_s}{\lambda} = 0$$

$$\tau_{\theta z} = -\frac{\partial M_{s\theta}}{\partial \bar{s}} - \frac{2}{\bar{s}}M_{s\theta} - \frac{1}{\bar{s}}\frac{\partial M_\theta}{\partial \bar{\theta}} + \frac{1}{\lambda}Q_\theta + \frac{m_2}{2\lambda}\ddot{v}$$
$$+ \frac{m_{21}}{\lambda}\ddot{\beta}_\theta - \frac{m_\theta}{\lambda} = 0$$

Note that we have neglected N_s, N_θ, and $N_{s\theta}$ in the third to sixth terms of τ_z because they are small when compared with N_s^i, N_θ^i, and $N_{s\theta}^i$. This is a good assumption for most engineering problems of this type.

Finally, we may write the equilibrium equations given by Eqs. (15) in terms of the displacements and rotations. On substituting Eqs. (3) into Eqs. (4), and further into Eqs. (15), we may express the equations of dynamic equilibrium in terms of the five displacements, namely,

$$L_{11}u + L_{12}v + L_{13}w + L_{14}\beta_s + L_{15}\beta_\theta$$
$$+ N_1(w) + \left(m_1/\lambda^2\right)\ddot{u} + \left(m_2/2\lambda^2\right)\ddot{\beta}_s - \left(q_s/\lambda^2\right) = 0$$

$$L_{21}u + L_{22}v + L_{23}w + L_{24}\beta_s + L_{25}\beta_\theta$$
$$+ N_2(w) + \left(m_1/\lambda^2\right)\ddot{v} + \left(m_2/2\lambda^2\right)\ddot{\beta}_\theta - \left(q_\theta/\lambda^2\right) = 0$$

$$L_{31}u + L_{32}v + L_{33}w + L_{34}\beta_s + L_{35}\beta_\theta$$
$$+ N_3(w) + \left(m_1/\lambda^2\right)\ddot{w} - \left(q_z/\lambda^2\right) = 0 \qquad (16)$$

$$L_{41}u + L_{42}v + L_{43}w + L_{44}\beta_s + L_{45}\beta_\theta$$
$$+ N_4(w) + \left(m_2/2\lambda^2\right)\ddot{u} + \left(m_{12}/\lambda^2\right)\ddot{\beta}_s - \left(m_s/\lambda^2\right) = 0$$

$$L_{51}u + L_{52}v + L_{53}w + L_{54}\beta_s + L_{55}\beta_\theta$$
$$+ N_5(w) + \left(m_2/2\lambda^2\right)\ddot{v} + \left(m_{21}/\lambda^2\right)\ddot{\beta}_\theta - \left(m_\theta/\lambda^2\right) = 0$$

where

$$L_{11} = -A_{11}\frac{\partial^2}{\partial \bar{s}^2} - \frac{A_{11}}{\bar{s}}\frac{\partial}{\partial \bar{s}} + \frac{A_{22}}{\bar{s}^2} - \frac{A_{33}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2}$$

$$L_{12} = -\frac{(A_{12} + A_{33})}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}} + \frac{(A_{22} + A_{33})}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}$$

$$L_{13} = \frac{A_{22}\cot\alpha}{\bar{s}^2} - \frac{A_{12}\cot\alpha}{\bar{s}}\frac{\partial}{\partial \bar{s}}$$

$$L_{14} = -B_{11}\frac{\partial^2}{\partial \bar{s}^2} - \frac{B_{11}}{\bar{s}}\frac{\partial}{\partial \bar{s}} + \frac{B_{22}}{\bar{s}^2} - \frac{B_{33}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2}$$

$$L_{15} = \frac{(B_{22} + B_{33})}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}} - \frac{(B_{12} + B_{33})}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}}$$

$$L_{21} = -\frac{(A_{12} + A_{33})}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}} - \frac{(A_{22} + A_{33})}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}$$

$$L_{22} = -A_{33}\left(\frac{\partial^2}{\partial \bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{s}} - \frac{1}{\bar{s}^2}\right) - \frac{A_{22}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2} + \frac{A_{44}\cot^2\alpha}{\bar{s}^2}$$

$$L_{23} = -\frac{(A_{22} + A_{44})\cot\alpha}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}$$

$$L_{24} = -\frac{(B_{12} + B_{33})}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}} - \frac{(B_{22} + B_{33})}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}$$

$$L_{25} = -B_{33}\left(\frac{\partial^2}{\partial \bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{s}} - \frac{1}{\bar{s}^2}\right) - \frac{B_{22}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2} - \frac{A_{44}\cot\alpha}{\bar{s}\lambda}$$

$$L_{31} = \frac{A_{12}\cot\alpha}{\bar{s}}\frac{\partial}{\partial \bar{s}} + \frac{A_{22}\cot\alpha}{\bar{s}^2} \qquad L_{32} = -L_{23}$$

$$L_{33} = -A_{55}\left(\frac{\partial^2}{\partial \bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{s}}\right) - \frac{A_{44}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2} - \frac{A_{22}\cot^2\alpha}{\bar{s}^2}$$
$$- \lambda\left(\frac{\partial N_s^i}{\partial \bar{s}} + \frac{N_s^i}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial N_{s\theta}^i}{\partial \bar{\theta}}\right)\frac{\partial}{\partial \bar{s}} - \lambda N_s^i\frac{\partial^2}{\partial \bar{s}^2} - \frac{2\lambda N_{s\theta}^i}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}}$$
$$- \lambda\left(\frac{1}{\bar{s}^2}\frac{\partial N_\theta^i}{\partial \bar{\theta}} + \frac{1}{\bar{s}}\frac{\partial N_{s\theta}^i}{\partial \bar{s}}\right)\frac{\partial}{\partial \bar{\theta}} - \frac{\lambda N_\theta^i}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2}$$

$$L_{34} = \left(\frac{B_{12}\cot\alpha}{\bar{s}} - \frac{A_{55}}{\lambda}\right)\frac{\partial}{\partial \bar{s}} - \frac{A_{55}}{\lambda\bar{s}} + \frac{B_{22}\cot\alpha}{\bar{s}^2}$$

$$L_{35} = \left(\frac{B_{22}\cot\alpha}{\bar{s}^2} - \frac{A_{44}}{\lambda\bar{s}}\right)\frac{\partial}{\partial \bar{\theta}}$$

$$L_{41} = L_{14} \qquad L_{42} = L_{15}$$

$$L_{43} = \left(\frac{A_{55}}{\lambda} - \frac{B_{12}\cot\alpha}{\bar{s}}\right)\frac{\partial}{\partial \bar{s}} + \frac{B_{22}\cot\alpha}{\bar{s}^2}$$

$$L_{44} = -D_{11}\left(\frac{\partial^2}{\partial \bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{s}}\right) + \frac{D_{22}}{\bar{s}^2} - \frac{D_{33}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2} + \frac{A_{55}}{\lambda^2}$$

$$L_{45} = \frac{(D_{22} + D_{33})}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}} - \frac{(D_{12} + D_{33})}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}}$$

$$L_{51} = L_{24} \qquad L_{52} = L_{25} \qquad L_{53} = -L_{35}$$

$$L_{54} = -\frac{(D_{12} + D_{33})}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}} - \frac{(D_{22} + D_{33})}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}$$

$$L_{55} = -D_{33}\left(\frac{\partial^2}{\partial \bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{s}} - \frac{1}{\bar{s}^2}\right) - \frac{D_{22}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2} + \frac{A_{44}}{\lambda^2} \qquad (17)$$

and the nonlinear terms in the equations of motion given by Eqs. (16) are

$$N_1(w) = -A_{11}\lambda\frac{\partial w}{\partial \bar{s}}\frac{\partial^2 w}{\partial \bar{s}^2} + (A_{12} + A_{22})\frac{\lambda}{2\bar{s}^3}\left(\frac{\partial w}{\partial \bar{\theta}}\right)^2$$
$$- (A_{12} + A_{33})\frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial \bar{\theta}}\frac{\partial^2 w}{\partial \bar{s}\partial \bar{\theta}} - (A_{11} - A_{12})\frac{\lambda}{2\bar{s}}\left(\frac{\partial w}{\partial \bar{s}}\right)^2$$
$$- A_{33}\frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial \bar{s}}\frac{\partial^2 w}{\partial \bar{\theta}^2}$$

$$N_2(w) = -(A_{12} + A_{33})\frac{\lambda}{\bar{s}}\frac{\partial w}{\partial \bar{s}}\frac{\partial^2 w}{\partial \bar{s}\partial \bar{\theta}} - A_{22}\frac{\lambda}{\bar{s}^3}\frac{\partial w}{\partial \bar{\theta}}\frac{\partial^2 w}{\partial \bar{\theta}^2}$$
$$- A_{33}\left(\frac{\lambda}{\bar{s}}\frac{\partial^2 w}{\partial \bar{s}^2}\frac{\partial w}{\partial \bar{\theta}} + \frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial \bar{s}}\frac{\partial w}{\partial \bar{\theta}}\right)$$

$$N_3(w) = A_{12}\frac{\lambda \cot\alpha}{2\bar{s}}\left(\frac{\partial w}{\partial \bar{s}}\right)^2 + A_{22}\frac{\lambda \cot\alpha}{2\bar{s}^3}\left(\frac{\partial w}{\partial \bar{\theta}}\right)^2$$

$$N_4(w) = -B_{11}\lambda\frac{\partial w}{\partial \bar{s}}\frac{\partial^2 w}{\partial \bar{s}^2} + (B_{12}+B_{22})\frac{\lambda}{2\bar{s}^3}\left(\frac{\partial w}{\partial \bar{\theta}}\right)^2 - (B_{12}+B_{33})\frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial \bar{\theta}}\frac{\partial^2 w}{\partial \bar{s}\partial \bar{\theta}} - (B_{11}-B_{12})\frac{\lambda}{2\bar{s}}\left(\frac{\partial w}{\partial \bar{s}}\right)^2 - B_{33}\frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial \bar{s}}\frac{\partial^2 w}{\partial \bar{\theta}^2}$$

$$N_5(w) = -(B_{12}+B_{33})\frac{\lambda}{\bar{s}}\frac{\partial w}{\partial \bar{s}}\frac{\partial^2 w}{\partial \bar{s}\partial \bar{\theta}} - B_{22}\frac{\lambda}{\bar{s}^3}\frac{\partial w}{\partial \bar{\theta}}\frac{\partial^2 w}{\partial \bar{\theta}^2} - B_{33}\left(\frac{\lambda}{\bar{s}}\frac{\partial^2 w}{\partial \bar{s}^2}\frac{\partial w}{\partial \bar{\theta}} + \frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial \bar{s}}\frac{\partial w}{\partial \bar{\theta}}\right) \tag{18}$$

The nonlinear equations can be extraordinarily difficult to solve. The equations raise mathematical problems in their own right. Fortunately, the nonlinear terms in Eqs. (16) may be neglected in most engineering problems.

Boundary and Initial Conditions

The related boundary conditions at both ends of the conical shell may be expressed in variational form as follows:

$$\delta J_b = \int_T \mathrm{d}t\left\{\oint_{C_f}\left[(N_s - N_s^*)\delta u + (N_{s\theta} - N_{s\theta}^*)\delta v + (Q_s - Q_s^*)\delta w + (M_s - M_s^*)\delta\beta_s + (M_{s\theta} - M_{s\theta}^*)\delta\beta_\theta\right]\frac{\bar{s}\,\mathrm{d}\bar{\theta}}{\lambda} + \oint_{C_d}\left[(u-u^*)\delta N_s + (v-v^*)\delta N_{s\theta} + (w-w^*)\delta Q_s + (\beta_s-\beta_s^*)\delta M_s + (\beta_\theta-\beta_\theta^*)\delta M_{s\theta}\right]\frac{\bar{s}\,\mathrm{d}\bar{\theta}}{\lambda}\right\} \tag{19}$$

where C_f is the boundary part that the forces are prescribed and C_d is the boundary part that the displacements are prescribed. The asterisk denotes the prescribed values of displacements and forces.

The initial conditions are given in the variational form as follows:

$$\delta J_i = \int_A \{[u(\bar{s},\bar{\theta},t_0) - u^*(\bar{s},\bar{\theta})]\delta\dot{u}(\bar{s},\bar{\theta}) + [v(\bar{s},\bar{\theta},t_0) - v^*(\bar{s},\bar{\theta})]\delta\dot{v}(\bar{s},\bar{\theta}) + [w(\bar{s},\bar{\theta},t_0) - w^*(\bar{s},\bar{\theta})]\delta\dot{w}(\bar{s},\bar{\theta}) + [\beta_s(\bar{s},\bar{\theta},t_0) - \beta_s^*(\bar{s},\bar{\theta})]\delta\dot{\beta}_s(\bar{s},\bar{\theta}) + [\beta_\theta(\bar{s},\bar{\theta},t_0) - \beta_\theta^*(\bar{s},\bar{\theta})]\delta\dot{\beta}_\theta(\bar{s},\bar{\theta}) + [\dot{u}(\bar{s},\bar{\theta},t_0) - \dot{u}^*(\bar{s},\bar{\theta})]\delta u(\bar{s},\bar{\theta}) + [\dot{v}(\bar{s},\bar{\theta},t_0) - \dot{v}^*(\bar{s},\bar{\theta})]\delta v(\bar{s},\bar{\theta}) + [\dot{w}(\bar{s},\bar{\theta},t_0) - \dot{w}^*(\bar{s},\bar{\theta})]\delta w(\bar{s},\bar{\theta}) + [\dot{\beta}_s(\bar{s},\bar{\theta},t_0) - \dot{\beta}_s^*(\bar{s},\bar{\theta})]\delta\beta_s(\bar{s},\bar{\theta}) + [\dot{\beta}_\theta(\bar{s},\bar{\theta},t_0) - \dot{\beta}_\theta^*(\bar{s},\bar{\theta})]\delta\beta_\theta(\bar{s},\bar{\theta})\}\,\mathrm{d}A \tag{20}$$

Generalized Variational Theorem

The fundamental equations of the conical shell may be stated with a generalized variational statement.[28] Now, let us define a functional J whose first variation is given by

$$\delta J = \delta J_e^d + \delta J_b + \delta J_k + \delta J_c + \delta J_i + \delta J_e^i + \delta J_b^i + \delta J_k^i + \delta J_c^i \tag{21}$$

with

$$\Lambda_c = (u, v, w, \beta_s, \beta_\theta \in C_{12}\ \varepsilon_s, \varepsilon_\theta, \varepsilon_{s\theta}, \varepsilon_{sz}, \varepsilon_{\theta z};\ \kappa_s, \kappa_\theta, \kappa_{s\theta} \in C_{00}\ \ N_s, N_\theta, N_{s\theta};\ M_s, M_\theta, M_{s\theta}, Q_s, Q_\theta \in C_{10}) \tag{22}$$

where J_b^i, J_k^i, and J_c^i are the corresponding functionals related with boundary conditions, kinematical relations, and constitutive relations for initial stresses, respectively. Then, of all of the admissible states Λ_c, only those that admit the functional J have zero first variation, if and only if they satisfy the equilibrium equations, the displacement-strain equations, the constitutive equations, the boundary conditions, and the initial conditions as appropriate Euler equations. The variational theorem $\delta J(\Lambda_c) = 0$ evidently generates, by the use of the fundamental lemma of the calculus of variations, the complete set of the basic equations of stiffened composite shell with initial stresses.

Uniqueness of Solutions

In the previous section, a set of two-dimensional, differential, approximate equations of Donnell's type is derived for the dynamic response of a laminated conical shell with stringers and rings. The two-dimensional governing equations of the stiffened laminated conical shell are constructed by the use of the virtual work principle within the limits of the well-known Kirchoff–Love hypotheses of thin shells. Now, the boundary and initial conditions are obtained, which are sufficient to ensure the uniqueness in solutions of the dynamical governing equations. Of the several arguments to be used to establish the uniqueness of solutions in elasticity,[29] the classical energy argument, which relies on the positive definiteness of kinetic and potential energies, is used in this study. Kirchoff[30] used the energy argument at establishing uniqueness in elastostatics, and so did Neumann[31] in elastodynamics and Weiner[32] in thermoelasticity. A uniqueness theorem of Neumann's type is proved for solutions of the initial mixed-boundary value problems defined by the two-dimensional governing equations of the stiffened cylindrical shell.[33,34] Mecitoglu and Dökmeci[10] examined the uniqueness in solutions of the dynamic governing equations of stiffened cylindrical shells.

To begin, consider two possible sets of solutions to the governing equations of the stiffened conical shell, namely,

$$\Lambda_\beta = (u, v, w, \beta_s, \beta_\theta \in C_{12}\ \varepsilon_s, \varepsilon_\theta, \varepsilon_{s\theta}, \varepsilon_{sz}, \varepsilon_{\theta z};\ \kappa_s, \kappa_\theta, \kappa_{s\theta} \in C_{00}\ \ N_s, N_\theta, N_{s\theta};\ M_s, M_\theta, M_{s\theta}, Q_s, Q_\theta \in C_{10})_\beta \qquad \beta = 1, 2 \tag{23}$$

Let the difference set of two solutions be denoted by $\Lambda = \Lambda_2 - \Lambda_1$. The difference set of solutions apparently satisfies all of the governing equations of the stiffened conical shell because of their linearity. It will be shown that the homogeneous linear governing equations possess only the zero solution; that is, the two sets of solutions (23) are equivalent under the pertinent boundary and initial conditions. In so doing, we introduce a relation of the form

$$\Gamma = \int_T (\Gamma_s + \Gamma_\theta + \Gamma_z + \Gamma_{sz} + \Gamma_{\theta z})\,\mathrm{d}t = 0 \tag{24}$$

with

$$\Gamma_s = \iint_A (\tau_s + \tau_s^i)\dot{u}\,\mathrm{d}A \qquad \Gamma_\theta = \iint_A (\tau_\theta + \tau_\theta^i)\dot{v}\,\mathrm{d}A$$
$$\Gamma_z = \iint_A (\tau_z + \tau_z^i)\dot{w}\,\mathrm{d}A \qquad \Gamma_{sz} = \iint_A (\tau_{sz} + \tau_{sz}^i)\dot{\beta}_s\,\mathrm{d}A \tag{25}$$
$$\Gamma_{\theta z} = \iint_A (\tau_{\theta z} + \tau_{\theta z}^i)\dot{\beta}_\theta\,\mathrm{d}A$$

where τ_s, τ_θ, τ_z, t_{sz}, and $\tau_{\theta z}$ are defined by Eqs. (15) and τ_s^i, τ_θ^i, τ_z^i, τ_{sz}^i, and $\tau_{s\theta}^i$ are defined by Eqs. (14).

Now, let us calculate the rates of the strain and kinetic energies of the stiffened conical shell in terms of the displacement components. The rate of the strain energy is expressed with respect to the difference set of the solutions in the form

$$\dot{U} = \iint_A \left[(N_s + N_s^i)\dot{\varepsilon}_s + (N_\theta + N_\theta^i)\dot{\varepsilon}_\theta + (N_{s\theta} + N_{s\theta}^i)\dot{\varepsilon}_{s\theta} + (M_s + M_s^i)\dot{\kappa}_s + (M_\theta + M_\theta^i)\dot{\kappa}_\theta + (M_{s\theta} + M_{s\theta}^i)\dot{\kappa}_{s\theta} + (Q_s + Q_s^i)\dot{\varepsilon}_{sz} + (Q_\theta + Q_\theta^i)\dot{\varepsilon}_{s\theta}\right]\mathrm{d}A \tag{26}$$

in terms of the difference set of the solutions. Using the strain-displacement equations (3) in Eq. (26) and integrating the equation by parts, one arrives at the rate of the form

$$
\begin{aligned}
\delta\dot{U} = \lambda \int\!\!\int_A \Bigg(\bigg\{ & -\frac{\partial}{\partial \bar{s}}(N_s + N_s^i) - \frac{1}{\bar{s}}[(N_s + N_s^i) - (N_\theta + N_\theta^i)] \\
& - \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{\theta}}(N_{s\theta} + N_{s\theta}^i)\bigg\}\dot{u} + \bigg[-\frac{\partial}{\partial \bar{s}}(N_{s\theta} + N_{s\theta}^i) \\
& - \frac{2}{\bar{s}}(N_{s\theta} + N_{s\theta}^i) - \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{\theta}}(N_\theta + N_\theta^i) - \frac{\cot\alpha}{\bar{s}}(Q_\theta + Q_\theta^i)\bigg]\dot{v} \\
& + \bigg\{-\frac{\partial}{\partial \bar{s}}(Q_s + Q_s^i) - \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{\theta}}(Q_\theta + Q_\theta^i) - \frac{1}{\bar{s}}(Q_s + Q_s^i) \\
& + \frac{\cot\alpha}{\bar{s}}(N_\theta + N_\theta^i) - \frac{\lambda}{\bar{s}}\frac{\partial}{\partial \bar{s}}\left[(N_s + N_s^i)\bar{s}\frac{\partial w}{\partial \bar{s}}\right] \\
& - \frac{\lambda}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}\left[(N_\theta + N_\theta^i)\frac{\partial w}{\partial \bar{\theta}}\right] - \frac{\lambda}{\bar{s}}\frac{\partial}{\partial \bar{s}}\left[(N_{s\theta} + N_{s\theta}^i)\frac{\partial w}{\partial \bar{\theta}}\right] \\
& - \frac{\partial}{\partial \bar{\theta}}\left[(N_{s\theta} + N_{s\theta}^i)\frac{\partial w}{\partial \bar{s}}\right]\bigg\}\dot{w} + \bigg\{-\frac{\partial}{\partial \bar{s}}(M_s + M_s^i) \\
& - \frac{1}{\bar{s}}[(M_s + M_s^i) - (M_\theta + M_\theta^i)] - \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{\theta}}(M_{s\theta} + M_{s\theta}^i) \\
& + \frac{1}{\lambda}(Q_s + Q_s^i)\bigg\}\dot{\beta}_s + \bigg[-\frac{\partial}{\partial \bar{s}}(M_{s\theta} + M_{s\theta}^i) - \frac{2}{\bar{s}}(M_{s\theta} + M_{s\theta}^i) \\
& - \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{\theta}}(M_\theta + M_\theta^i) + \frac{1}{\lambda}(Q_\theta + Q_\theta^i)\bigg]\dot{\beta}_\theta\Bigg)\,\mathrm{d}A \\
& + \oint_C \psi_\theta \bar{s}\,\mathrm{d}\bar{\theta} + \int_L \psi_s\,\mathrm{d}\bar{s}
\end{aligned}
\tag{27}
$$

In this equation, the quantities of the form

$$
\begin{aligned}
\psi_s = \frac{1}{\lambda}\bigg\{ & (N_{s\theta} + N_{s\theta}^i)\dot{u} + (N_\theta + N_\theta^i)\dot{v} \\
& + \left[(Q_\theta + Q_\theta^i) + \frac{\lambda}{\bar{s}}(N_\theta + N_\theta^i)\frac{\partial w}{\partial \bar{\theta}} + \lambda(N_{s\theta} + N_{s\theta}^i)\frac{\partial w}{\partial \bar{s}}\right]\dot{w} \\
& + (M_{s\theta} + M_{s\theta}^i)\dot{\beta}_s + (M_\theta + M_\theta^i)\dot{\beta}_\theta\bigg\}\bigg|_{\bar{\theta}=0}^{2\pi\sin\alpha}
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
\psi_\theta = \frac{1}{\lambda}\bigg\{ & (N_s + N_s^i)\dot{u} + (N_{s\theta} + N_{s\theta}^i)\dot{v} \\
& + \left[(Q_s + Q_s^i) + \frac{\lambda}{\bar{s}}(N_{s\theta} + N_{s\theta}^i)\frac{\partial w}{\partial \bar{\theta}}\right]\dot{w} \\
& + (M_s + M_s^i)\dot{\beta}_s + (M_{s\theta} + M_{s\theta}^i)\bigg\}\bigg|_{\bar{s}=\bar{s}_1}^{1}
\end{aligned}
\tag{29}
$$

are introduced.

Likewise, the rate of the total kinetic energy may be written as follows:

$$
\begin{aligned}
\dot{K} = \int\!\!\int_A \bigg[& m_1(\dot{u}\ddot{u} + \dot{v}\ddot{v} + \dot{w}\ddot{w}) + \frac{m_2}{2}(\dot{u}\ddot{\beta}_s + \dot{\beta}_s\ddot{u} + \dot{v}\ddot{\beta}_\theta + \dot{\beta}_\theta\ddot{v}) \\
& + m_{12}\dot{\beta}_s\ddot{\beta}_s + m_{21}\dot{\beta}_\theta\ddot{\beta}_\theta\bigg]\,\mathrm{d}A
\end{aligned}
\tag{30}
$$

in terms of the difference set of the solutions. Integrating this equation by parts, one arrives at the rate of the form

$$
\begin{aligned}
\dot{K} = \int\!\!\int_A \bigg[& \left(m_1\ddot{u} + \frac{1}{2}m_2\ddot{\beta}_s\right)\dot{u} + \left(m_1\ddot{v} + \frac{1}{2}m_2\ddot{\beta}_\theta\right)\dot{v} + m_1\ddot{w} \\
& + \left(\frac{1}{2}m_2\ddot{u} + m_{12}\ddot{\beta}_s\right)\dot{\beta}_s + \left(\frac{1}{2}m_2\ddot{v} + m_{21}\ddot{\beta}_\theta\right)\dot{\beta}_\theta\bigg]\,\mathrm{d}A
\end{aligned}
\tag{31}
$$

In view of the energy rates (27) and (31), the relation (24) is expressed as

$$
\Gamma = \int_T (\dot{U} + \dot{K})\,\mathrm{d}t - \int_T \mathrm{d}t\left(\oint_C \psi_\theta \bar{s}\,\mathrm{d}\bar{\theta} + \int_L \psi_s\,\mathrm{d}\bar{s}\right) = 0 \tag{32}
$$

An integration of this equation with respect to time yields

$$
K(t_1) + U(t_1) = K(t_0) + U(t_0) + \int_T \psi\,\mathrm{d}t \tag{33}
$$

with

$$
\psi = \oint_C \psi_\theta \bar{s}\,\mathrm{d}\bar{\theta} + \int_L \psi_s\,\mathrm{d}\bar{s} \tag{34}
$$

The kinetic and strain energy densities are positive-definite, by definition, and initially zero, so that the total kinetic energy and strain energy, K and U, calculated by integration from the difference set of solutions for the stiffened conical elastic shell have the same properties. Thus, it follows from Eq. (32) that

$$
K(t_1) = U(t_1) = K(t_0) = U(t_0) = 0 \tag{35}
$$

This implies a trivial solution for the difference set of solutions, Λ_c, because the remaining term ψ of Eq. (32) vanishes in view of Eq. (20). Hence, the uniqueness is ensured in solutions of the governing equations of the stiffened conical elastic shell. The following theorem is then concluded.

Theorem. Given the regular region of a stiffened cylindrical elastic shell in the Euclidean three-dimensional space, then there exists at most one set of single-valued solutions Λ_c, namely,

$$
\begin{aligned}
\Lambda_c = (& u, v, w, \beta_s, \beta_\theta \in C_{12}\ \varepsilon_s, \varepsilon_\theta, \varepsilon_{s\theta}, \varepsilon_{sz}, \varepsilon_{\theta z}; \kappa_s, \kappa_\theta, \kappa_{s\theta} \in C_{00} \\
& N_s, N_\theta, N_{s\theta}; M_s, M_\theta, M_{s\theta}, Q_s, Q_\theta \in C_{10})
\end{aligned}
$$

which satisfies all of the governing equations of the stiffened shell, provided that kinetic and strain energies are positive-definite and boundary (19) and initial conditions (20) are prescribed. The term C_{mn} refers to the function with derivatives of order up to and including (m) and (n) with respect to space coordinates $(\bar{s}, \bar{\theta})$ and time t.

Numerical Example

As an illustrative example, we study the linear free vibrations of a stiffened composite laminated conical shell without the influence of initial stresses numerically. The equations of motion given by Eq. (16) are solved by assuming displacement functions of the form

$$
\begin{aligned}
u(\bar{s}, \bar{\theta}, t) &= \sum_{n=1}^{\infty} \hat{u}(\bar{s}) \cos\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t \\
v(\bar{s}, \bar{\theta}, t) &= \sum_{n=1}^{\infty} \hat{v}(\bar{s}) \sin\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t \\
w(\bar{s}, \bar{\theta}, t) &= \sum_{n=1}^{\infty} \hat{w}(\bar{s}) \cos\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t \\
\beta_s(\bar{s}, \bar{\theta}, t) &= \sum_{n=1}^{\infty} \hat{\beta}_s(\bar{s}) \cos\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t \\
\beta_\theta(\bar{s}, \bar{\theta}, t) &= \sum_{n=1}^{\infty} \hat{\beta}_\theta(\bar{s}) \sin\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t
\end{aligned}
\tag{36}
$$

with the harmonic motion assumption. Because the shell is axisymmetric (e.g., no cutouts) and has axisymmetric boundary conditions, the vibration modes uncouple with respect to θ and the summations on the n are not considered in Eqs. (36). Substituting these equations into Eqs. (16) and canceling the initial stresses, nonlinear terms, and dynamic external loads, we obtain five ordinary differential equations:

$$-A_{11}\hat{u}'' - \frac{A_{11}}{\bar{s}}\hat{u}' + \frac{A_{22}+A_{33}\psi^2}{\bar{s}^2}\hat{u} - \frac{A_{12}+A_{33}}{\bar{s}}\psi\hat{v}' + \frac{A_{22}+A_{33}}{\bar{s}^2}\psi\hat{v} - \frac{A_{12}\cot\alpha}{\bar{s}}\hat{w}' + \frac{A_{22}\cot\alpha}{\bar{s}^2}\hat{w} - B_{11}\hat{\beta}_s'' - \frac{B_{11}}{\bar{s}}\hat{\beta}_s' + \frac{B_{22}+B_{33}\psi^2}{\bar{s}^2}\hat{\beta}_s - \frac{B_{12}+B_{33}}{\bar{s}}\psi\hat{\beta}_\theta' + \frac{B_{22}+B_{33}}{\bar{s}^2}\psi\hat{\beta}_\theta = \frac{\omega^2}{\lambda^2}\left(m_1\hat{u} + \frac{1}{2}m_2\hat{\beta}_s\right)$$

$$\frac{A_{12}+A_{33}}{\bar{s}}\psi\hat{u}' + \frac{A_{22}+A_{33}}{\bar{s}^2}\psi\hat{u} - A_{33}\hat{v}'' - A_{33}\frac{1}{\bar{s}}\hat{v}' + \frac{A_{22}\psi^2+A_{33}+A_{44}\cot^2\alpha}{\bar{s}^2}\hat{v} + \frac{(A_{22}+A_{44})\cot\alpha}{\bar{s}^2}\psi\hat{w} + \frac{B_{12}+B_{33}}{\bar{s}}\psi\hat{\beta}_s' + \frac{B_{22}+B_{33}}{\bar{s}^2}\psi\hat{\beta}_s - B_{33}\hat{\beta}_\theta'' - B_{33}\frac{1}{\bar{s}}\hat{\beta}_\theta' + \left(\frac{B_{22}\psi^2+B_{33}}{\bar{s}^2} - \frac{A_{44}\cot\alpha}{\bar{s}\lambda}\right)\hat{\beta}_\theta = \frac{\omega^2}{\lambda^2}\left(m_1\hat{v} + \frac{1}{2}m_2\hat{\beta}_\theta\right)$$

$$\frac{A_{12}\cot\alpha}{\bar{s}}\hat{u}' + \frac{A_{22}\cot\alpha}{\bar{s}^2}\hat{u} + \frac{(A_{22}+A_{44})\cot\alpha}{\bar{s}^2}\psi\hat{v} - A_{55}\hat{w}'' - A_{55}\frac{1}{\bar{s}}\hat{w}' + \frac{A_{44}\psi^2 - A_{22}\cot^2\alpha}{\bar{s}^2}\hat{w} + \left(\frac{B_{12}\cot\alpha}{\bar{s}} - \frac{A_{55}}{\lambda}\right)\hat{\beta}_s' + \left(\frac{B_{22}\cot\alpha}{\bar{s}^2} - \frac{A_{55}}{\lambda\bar{s}}\right)\hat{\beta}_s + \left(\frac{B_{22}\cot\alpha}{\bar{s}^2} - \frac{A_{44}}{\lambda\bar{s}}\right)\psi\hat{\beta}_\theta = \frac{\omega^2}{\lambda^2}m_1\hat{w} \tag{37}$$

$$-B_{11}\hat{u}'' - \frac{B_{11}}{\bar{s}}\hat{u}' + \frac{B_{22}+B_{33}\psi^2}{\bar{s}^2}\hat{u} - \frac{B_{12}+B_{33}}{\bar{s}}\psi\hat{v}' + \frac{B_{22}+B_{33}}{\bar{s}^2}\psi\hat{v} + \left(\frac{A_{55}}{\lambda} - \frac{B_{12}\cot\alpha}{\bar{s}}\right)\hat{w}' + \frac{B_{22}\cot\alpha}{\bar{s}^2}\hat{w} - D_{11}\left(\hat{\beta}_s'' + \frac{1}{\bar{s}}\hat{\beta}_s'\right) + \left(\frac{D_{22}+D_{33}\psi^2}{\bar{s}^2} + \frac{A_{55}}{\lambda^2}\right)\hat{\beta}_s - \frac{D_{12}+D_{33}}{\bar{s}}\psi\hat{\beta}_\theta' + \frac{D_{22}+D_{33}}{\bar{s}^2}\psi\hat{\beta}_\theta = \frac{\omega^2}{\lambda^2}\left(m_{12}\hat{\beta}_s + \frac{1}{2}m_2\hat{u}\right)$$

$$\frac{B_{12}+B_{33}}{\bar{s}}\psi\hat{u}' + \frac{B_{22}+B_{33}}{\bar{s}^2}\psi\hat{u} - B_{33}\left(\hat{v}'' + \frac{1}{\bar{s}}\hat{v}'\right) + \left(\frac{B_{22}\psi^2+B_{33}}{\bar{s}^2} - \frac{A_{44}\cot\alpha}{\bar{s}\lambda}\right)\hat{v} + \left(\frac{B_{22}\cot\alpha}{\bar{s}^2} - \frac{A_{44}}{\lambda\bar{s}}\right)\psi\hat{w} + \frac{D_{12}+D_{33}}{\bar{s}}\psi\hat{\beta}_s' + \frac{D_{22}+D_{33}}{\bar{s}^2}\psi\hat{\beta}_s - D_{33}\left(\hat{\beta}_\theta'' + \frac{1}{\bar{s}}\hat{\beta}_\theta'\right) + \left(\frac{D_{22}\psi^2+D_{33}}{\bar{s}^2} + \frac{A_{44}}{\lambda^2}\right)\hat{\beta}_\theta = \frac{\omega^2}{\lambda^2}\left(m_{21}\hat{\beta}_\theta + \frac{1}{2}m_2\hat{v}\right)$$

where the prime denotes an ordinary derivative with respect to $\bar{s}$ and the parameter ψ is defined as $\psi = n/\sin\alpha$.

The finite difference method is applied to solve resulting ordinary differential equations together with the boundary conditions at the small end:

$$u = v = w = \beta_s = M_{s\theta} = 0 \tag{38a}$$

and at the large end:

$$N_s = v = w = M_s = M_{s\theta} = 0 \tag{38b}$$

The central difference formulas are used to obtain the derivatives with respect to $\bar{s}$.

The axisymmetric vibration of antisymmetric cross-plied laminated cones is considered in the illustrative example, and the material parameters of each layer are taken to be

$$E_s = 15E_\theta \qquad \nu_{s\theta} = 0.25 \qquad \nu_{sz} = \nu_{\theta z} = 0.3 \tag{39}$$

Thus the coefficients in Eqs. (37) are

$$\{A_{11}, A_{12}, A_{22}, A_{33}, A_{44}, A_{55}\} = h\left\{\tfrac{1}{2}(Q_{11}+Q_{22}), Q_{12}, \tfrac{1}{2}(Q_{11}+Q_{22}), Q_{33}, \tfrac{5}{6}Q_{44}, \tfrac{5}{6}Q_{55}\right\}$$
$$\{B_{11}, B_{12}, B_{22}, B_{33}\} = h^2\left\{\tfrac{1}{8}(Q_{11}-Q_{22}), 0, \tfrac{1}{8}(Q_{22}-Q_{11}), 0\right\}$$
$$\{D_{11}, D_{12}, D_{22}, D_{33}\} = h^3\left\{\tfrac{1}{24}(Q_{11}+Q_{22}), \tfrac{1}{12}Q_{12}, \tfrac{1}{24}(Q_{11}+Q_{22}), \tfrac{1}{12}Q_{33}\right\} \tag{40}$$

where h denotes the total thickness of the cone, and

$$\{Q_{11}, Q_{12}, Q_{22}\} = \frac{1}{1-\nu_{s\theta}\nu_{\theta s}}\{E_s, \nu_{s\theta}E_\theta, E_\theta\}$$
$$\{Q_{33}, Q_{44}, Q_{55}\} = E_\theta\left\{\frac{1}{2}, \frac{1}{2(1+\nu_{\theta z})}, \frac{1}{2(1+\nu_{sz})}\right\} \tag{41}$$

The geometrical properties of the truncated conical shell are taken to be $L/R = 0.5$ and $h/R = 0.01$. The following stiffening parameters are used for the stiffened shell case: $E_S = E_R = 0.52E_s$, $\rho_S = \rho_R = 1.73\rho_c$, $h_S = h_R = 5h$, $b_S = b_R = h$, and $N_S = N_R = 6$. Here, h_S, b_S, and N_S are the depth, width, and the number of stringers, respectively. The subscript R denotes the properties of a ring. Before presenting the results, let us introduce a frequency parameter defined as

$$\Omega = \omega R\sqrt{\rho_c h/A_{11}} \tag{42}$$

where A_{11} is determined for the unstiffened conical shell.

Table 1 shows the convergence of the results with the number of interior grid points. In this example, the semivertex angle α is taken to be 45 deg. The 30 grid points give good convergence. n denotes the number of circumferential waves.

Table 2 gives the fundamental frequency of the laminated truncated conical shell with and without stiffeners. The results of this analysis are compared with those obtained from the power series approach given by Tong[24] for the unstiffened laminated conical shell. Table 2 shows a good agreement with the results from Tong. The fundamental frequency of the laminated truncated conical shell decreases with the isotropic stringers and rings. The reason for the reducing effect of the stiffeners on the natural frequencies of the conical shell can be explained as the relative contribution of the stiffeners to the mass of the laminated conical shell being much more than the contribution of the stiffeners to the stiffness of the conical shell. To prove the preceding statement, let us define a parameter as the ratio of characteristic stiffness and mass terms, $P = A_{11}/m_1$. The ratio

Table 1 Convergence of the frequency parameters Ω (α = 45 deg)

	Number of grid points					
n	2	4	8	16	30	40
1	0.2083	0.2141	0.2165	0.2174	0.2176	0.2177
2	0.2201	0.2253	0.2275	0.2283	0.2285	0.2285
3	0.2503	0.2083	0.2568	0.2576	0.2579	0.2577
4	0.2894	0.2936	0.2953	0.2959	0.2961	0.2962

Table 2 **Fundamental frequency of the laminated composite conical shell with and without the stiffeners; α is the semivertex angle of the conical shell**

	Unstiffened shell		Stiffened shell	
α, deg	This analysis	Power series of Tong[24]	Internal stiffeners	External stiffeners
30	0.1989	0.1768	0.1395	0.1513
45	0.2176	0.2270	0.1474	0.1504
60	0.2136	0.2231	0.1437	0.1478

of the parameter of the unstiffened cone to that of the stiffened cone with the same geometric properties is found to be approximately 2.

Conclusions

In this study, all of the dynamic equations of a composite laminated conical thin shell reinforced by stringers and rings are derived in the variational form, and the uniqueness is examined in solutions of the governing shell equations. The dynamic equations of a stiffened composite laminated conical shell are based on the Donnell–Mushtari theory of elastic thin shells. A generalized variational theorem is given so as to describe the complete set of the fundamental equations of the conical shell. The geometric nonlinearities and initial stresses are taken into account in the derivation of governing equations. The rings and/or stringers are smeared out along the conical shell. The inhomogeneity of material properties is considered in the constitutive equations. The uniqueness is examined in solutions of the dynamic governing equations of stiffened shells, and a theorem of uniqueness is given that enumerates the initial and boundary conditions sufficient for the uniqueness. The dynamic equations are solved by the finite difference method to obtain the free vibration characteristics of a composite laminated shell and a good agreement is obtained with certain earlier results.

In the uniqueness theorem, the boundary and initial conditions that render ψ to zero are shown to be sufficient for the uniqueness in solutions of the governing equations of the stiffened conical elastic shell. The conditions (19) and (20), and also, to specify one member of each product in ψ of Eq. (34), ensure a unique solution for the governing equations. Besides, the sufficient conditions can be expressed in terms of the stress resultants as well as the displacement components, and they can be obtained by logarithmic convexity arguments with no restrictions on the positive-definiteness of energies.

In the illustrative example the free vibration of the stiffened laminated composite conical shell is studied numerically by the finite difference method. The isotropic stiffeners reduce the fundamental frequencies of the laminated conical shell with all of the semivertex angles, because the mass contribution of the stiffeners is greater than the stiffness contribution of them.

Finally, it is concluded that the present method can be employed to establish the dynamic governing equations of a general stiffened laminated shell and to prove the uniqueness theorem for the solution of the equations. A numerical and parametric study can be performed by accounting for the complicating effects such as initial stresses, inhomogeneity, and nonlinearity. The dynamic behavior of the laminated conical shells with temperature-dependent material properties under the aerothermal loads has a great importance for aerospace structures, and this will be a topic of future study.

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